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Supersymmetry and the Multi-Instanton Measure II. From $N=4$ to $N=0$

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Extending recent $N=1$ and $N=2$ results, we propose an explicit formula for the integration measure on the moduli space of charge-n ADHM multi-instantons in $N=4$ supersymmetric $SU(2)$ gauge theory. As a consistency check, we derive a renormalization group relation between the $N=4$, $N=2$, and $N=1$ measures. We then use this relation to construct the purely bosonic ("$N=0$") measure as well, in the classical approximation in which the one-loop small-fluctuations determinants is not included.

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1. Introduction

In a recent paper [1] (henceforth (I)) we constructed the collective coordinate integration measure for charge-n ADHM multi-instantons [2] in both $N=1$ and $N=2$ supersymmetric $SU(2)$ gauge theories. Here we present the analogous formula for the $N=4$ theory. As a nontrivial consistency check, we derive a renormalization group (RG) relation between the $N=4$ and $N=2$ measures, and between the $N=2$ and $N=1$ measures, that emerges when the appropriate components of the supermultiplets are given a mass which is taken to infinity. In turn, this RG relation also yields an interesting formula for the purely bosonic ("$N=0$") ADHM measure. However, unlike the supersymmetric cases, this $N=0$ measure is valid at the classical level only, i.e., excluding the one-loop small-fluctuations 't Hooft determinants over positive-frequency gauge and ghost modes in the self-dual background [3]. (It is not necessary to invoke this "classical approximation" in the $N=1,2,4$ cases, as the 't Hooft determinants cancel between bosonic and fermionic excitations so long as there is at least one supersymmetry [4].) Since substantial progress has been made towards the calculation of these determinants in the ADHM background [5,6], there is reason for optimism that our field-theoretic understanding of the multi-instanton sector in the $N=0$ model will come to match our current understanding of the single-instanton sector.

In what follows we will focus on pure $N=0,1,2$ or 4 supersymmetric gauge theories; the incorporation of additional matter in the fundamental representation of the gauge group is straightforward (see Sec. 4 of (I)). For general topological number n, the $N=1$ collective coordinate integration measure $d\mu^{(n)}_{\rm phys}$ is given in Eqs. (2.23) and (2.54) of (I):[1]

$$\begin{aligned}\int d\mu^{(n)}_{\rm phys} \;=\;& \frac{(C_1)^n}{{\rm Vol}\big(O(n)\big)}\int\prod_{i=1}^n d^4w_i d^2\mu_i \prod_{(ij)_n} d^4a'_{ij}d^2\mathcal{M}'_{ij} \\ &\times \prod_{\langle ij\rangle_n}\prod_{c=1,2,3}\delta\big(\tfrac14{\rm tr}_2\,\tau^c[(\bar a a)_{i,j}-(\bar a a)_{j,i}]\big)\,\delta^2\big((\bar a\mathcal{M})_{i,j}-(\bar a\mathcal{M})_{j,i}\big)\ .\end{aligned} \tag{1}$$

Here

$$a_{\alpha\dot\alpha} \;=\; \begin{pmatrix} w_{1\alpha\dot\alpha} & \cdots & w_{n\alpha\dot\alpha} \\ & a'_{\alpha\dot\alpha} & \end{pmatrix}\quad,\qquad \mathcal{M}^\gamma \;=\; \begin{pmatrix} \mu_1^\gamma & \cdots & \mu_n^\gamma \\ & \mathcal{M}'^\gamma & \end{pmatrix} \tag{2}$$

are, respectively, $(n+1)\times n$ quaternion-valued and Weyl-spinor-valued collective coordinate matrices describing $8n$ independent bosonic (gauge field) and $4n$ independent fermionic

[1] The ADHM and SUSY notation and conventions are as in (I). In particular $(ij)_n$ and $\langle ij\rangle_n$ stand for the ordered pairs (i,j) subject to $1\le i\le j\le n$ and $1\le i<j\le n$, respectively. Also see (I) for references to the earlier literature.

(gaugino) degrees of freedom of the super-multi-instanton. These matrices are subject to the symmetry conditions $a' = a'^T$ and $\mathcal{M}' = \mathcal{M}'^T$ as well as to the supersymmetrized ADHM constraints [2,7] implemented by the δ-functions in (1) (which are absent in the single-instanton sector, $n = 1$). Also C_1 is 't Hooft's 1-instanton factor [3]

$$C_1 \ = \ 2^9\, \Lambda^6_{N=1} \ \propto \ \exp(-8\pi^2/g^2_{N=1}) \tag{3}$$

where $\Lambda_{N=1}$ is the dynamically generated scale in the Pauli-Villars (PV) scheme, which is the natural scheme for instanton calculations [3].

Since the δ-functions in Eq. (1) are dictated by the ADHM formalism, and since, as shown in (I), the resulting measure turns out to be a supersymmetry invariant and also has the correct transformation property under the anomalous $U(1)_R$ symmetry, we made the stronger claim in (I) that this Ansatz is in fact unique. To see why, let us consider including an additional function of the collective coordinates, $f(a, \mathcal{M})$, in the integrand of Eq. (1). To preserve supersymmetry, we can require that f be a supersymmetry invariant. It is a fact that any non-constant function that is a supersymmetry invariant must contain fermion bilinear pieces (and possibly higher powers of fermions as well). By the rules of Grassmann integration, such bilinears would necessarily lift some of the adjoint fermion zero modes contained in $\mathcal{M}$. But since Eq. (1) contains precisely the right number of unlifted fermion zero modes dictated by the $U(1)_R$ anomaly, namely $4n$, this argument rules out the existence of a non-constant function f. Moreover, any constant f would be absorbed into the overall multiplicative factor, which is fixed by cluster decomposition as detailed in (I). In fact, a similar uniqueness argument applies to our proposed ADHM measure for $N = 2$ theories. Unfortunately, as we discuss below, the above argument cannot be applied directly to the $N = 4$ model where the anomaly vanishes.

In the $N = 2$ model the story is slightly more complicated due to the presence of an adjoint scalar field. However, in the absence of a VEV for this field, the anomaly dictates a total of $8n$ unlifted modes. The appropriate measure can then be determined by the same considerations as for the $N = 1$ case, including the above uniqueness argument. The $N = 2$ formula is given in Eqs. (3.19) and (3.27) of (I):

$$\begin{aligned}\int d\mu^{(n)}_{\rm phys} \ &= \ \frac{(C_1')^n}{{\rm Vol}\big(O(n)\big)} \int \prod_{i=1}^n d^4 w_i d^2\mu_i d^2\nu_i \prod_{(ij)_n} d^4 a'_{ij} d^2\mathcal{M}'_{ij} d^2\mathcal{N}'_{ij} \\ &\quad \times \prod_{\langle ij\rangle_n} \prod_{c=1,2,3} \delta\big(\tfrac14 {\rm tr}_2\, \tau^c[(\bar a a)_{i,j} - (\bar a a)_{j,i}]\big)\, \delta^2\big((\bar a\mathcal{M})_{i,j} - (\bar a\mathcal{M})_{j,i}\big) \\ &\quad \times \ \delta^2\big((\bar a\mathcal{N})_{i,j} - (\bar a\mathcal{N})_{j,i}\big)\, (\det \mathbf{L})^{-1}. \end{aligned} \tag{4}$$

Here $\mathcal{N}$, like $\mathcal{M}$, is a Weyl-spinor-valued matrix containing $4n$ independent adjoint Higgsino degrees of freedom; $\mathbf{L}$ is a certain $\frac{1}{2}n(n-1)\times\frac{1}{2}n(n-1)$ linear operator on the space of $n\times n$ antisymmetric matrices (see Eq. (3.10) of (I)); and the 1-instanton factor is [3]

$$C_1' \;=\; 2^8\,\pi^{-4}\,\Lambda^4_{N=2}\;\propto\;\exp(-8\pi^2/g^2_{N=2}) \tag{5}$$

with $\Lambda_{N=2}$ in the PV scheme as before.

The main application for the above measure is in the calculation of instanton corrections to the Coulomb branch of the $N=2$ theory, where the VEV of the adjoint scalar spontaneously breaks the gauge group down to an abelian subgroup. In the presence of a VEV, the $U(1)_R$ symmetry is also spontaneously broken which spoils the naive fermion zero mode counting implied by the anomaly. In addition, the instanton is no longer an exact solution of the equations of motion. These two features lead to an instanton action which depends explicitly on bosonic and fermionic collective coordinates, the latter dependence lifting all but four of the $8n$ fermion zero modes. This case is analyzed in detail in [8] using the constrained instanton of Affleck, Dine and Seiberg [9]. An important feature of the constrained instanton approach is that, at leading semiclassical order, the effects of a non-zero VEV enter *only* through the instanton action and hence there are no additional modifications to the measure (4). The relevant multi-instanton actions for a variety of $N=2$ supersymmetric models are assembled in Sec. 5 of (I).

2. Ansatz for the $N=4$ measure

Next we turn to the $N=4$ case, which is reviewed in Ref. [10]. In studying this model it is convenient to relabel $\mathcal{M}\to\mathcal{M}^1$ and $\mathcal{N}\to\mathcal{M}^2$ as in [10]. The $N=4$ model requires two additional adjoint fermion multiplets (adjoint Higgsinos), parametrized by collective coordinate matrices $\mathcal{M}^3$ and $\mathcal{M}^4$. An $SU(4)_R$ symmetry acts on these superscripts. The multi-instanton action for $N=4$ supersymmetric $SU(2)$ gauge theory then reads [10]:

$$\begin{aligned}
S^{N=4}_{\rm inst} \;=\;& 16\pi^2|\mathcal{A}_{00}|^2\sum_{k=1}^n|w_k|^2\;-\;8\pi^2\,{\rm Tr}_n\big(\mathcal{A}'\cdot\bar\Lambda+\mathcal{A}'_f(\mathcal{M}^1,\mathcal{M}^2)\cdot\bar\Lambda-\mathcal{A}'_f(\mathcal{M}^3,\mathcal{M}^4)\cdot\Lambda\big)\\
&+\;4\sqrt2\,\pi^2\sum_{k=1}^n\big(\mu_k^{1\alpha}\,\bar{\mathcal{A}}_{00\alpha}{}^{\beta}\,\mu^2_{k\beta}\;-\;\mu_k^{3\alpha}\,\mathcal{A}_{00\alpha}{}^{\beta}\,\mu^4_{k\beta}\big)\\
&+\;\pi^2\sum_{A,B,C,D=1}^4\epsilon_{ABCD}{\rm Tr}_n\,\mathcal{A}'_f(\mathcal{M}^A,\mathcal{M}^B)\cdot\Lambda_f(\mathcal{M}^C,\mathcal{M}^D)\;.
\end{aligned} \tag{6}$$

Here $\mathcal{A}_{00}$ is the $SU(2)$-valued VEV, which we have chosen to live in the Higgs which is the $N=1$ superpartner of the $\mathcal{M}^1$ Higgsino; the collective coordinates w_k and μ_k^A are as in

Eq. (2); and Λ and $\Lambda_f(\mathcal{M}^A,\mathcal{M}^B)$ are as in (I). Also $\mathcal{A}'$ and $\mathcal{A}'_f(\mathcal{M}^A,\mathcal{M}^B)$ are defined as the solutions to $\mathbf{L}\cdot\mathcal{A}'=\Lambda$ and $\mathbf{L}\cdot\mathcal{A}'_f(\mathcal{M}^A,\mathcal{M}^B)=\Lambda_f(\mathcal{M}^A,\mathcal{M}^B)$.

We now discuss the $N=4$ collective coordinate integration measure. As in the $N=1$ and $N=2$ cases we seek an Ansatz with the following four properties:

(i) Invariance under $N=4$ supersymmetry;

(ii) Invariance under the internal $O(n)$ transformations which are redundant degrees of freedom endemic to the ADHM construction;

(iii) cluster decomposition in the dilute-gas limit of large space-time separation between instantons;

(iv) agreement with known formulae in the 1-instanton sector.

We will show that the following expression embodies these properties:

$$\begin{aligned}\int d\mu^{(n)}_{\rm phys} \;=\; &\frac{(C''_1)^n}{{\rm Vol}\big(O(n)\big)}\int\prod_{A=1,2,3,4}\prod_{i=1}^{n}d^4w_i\,d^2\mu^A_i\prod_{(ij)_n}d^4a'_{ij}\,d^2\mathcal{M}^{A\prime}_{ij}\\ &\times\prod_{\langle ij\rangle_n}\prod_{c=1,2,3}\delta\big(\tfrac14{\rm tr}_2\,\tau^c[(\bar a a)_{i,j}-(\bar a a)_{j,i}]\big)\prod_{A=1,2,3,4}\delta^2\big((\bar a\mathcal{M}^A)_{i,j}-(\bar a\mathcal{M}^A)_{j,i}\big)\\ &\qquad\times\;(\det\mathbf{L})^{-3}\;,\end{aligned}\tag{7}$$

where C''_1 is the 1-instanton factor, again in the PV scheme [3]:

$$C''_1\;=\;2^6\,\pi^{-12}\,\exp(-8\pi^2/g^2_{N=4})\;.\tag{8}$$

Note that there is no dynamically generated scale in the $N=4$ model as it is conformally invariant, and finite (albeit scheme dependent due to cancellations between individually divergent diagrams). In the 1-instanton sector the second and third lines of Eq. (7) are omitted, as in the $N=1$ and $N=2$ cases.

Before verifying properties (i)-(iv), we remind the reader that the $N=1$ and $N=2$ measures also needed to satisfy a fifth defining property discussed in (I): the number of adjoint fermion zero modes left unsaturated by the measure had to equal $4n$ and $8n$, respectively. This fermionic mode counting is dictated by the anomaly, and is at the heart of the uniqueness argument given above. But in the $N=4$ model the anomaly vanishes, and the issue of fermionic mode counting is less definite. To see this ambiguity, note that the Ansatz (7) leaves $16n$ Grassmann modes unlifted, i.e., twice the $N=2$ counting as one might expect. On the other hand it appears to us to be purely a matter of convention whether or not the exponentiated action (6) should be considered part of the measure, especially in the limit of vanishing VEV. And here the $N=4$ action differs in a significant way from its $N=1$ and $N=2$ counterparts: in this limit the action remains nontrivial,

specifically the last term in Eq. (6) survives. If one then chooses to include this fermion quadrilinear term

$$\exp\big(-\pi^2 \sum_{A,B,C,D=1}^{4} \epsilon_{ABCD} \mathrm{Tr}_n\, \mathcal{A}'_f(\mathcal{M}^A, \mathcal{M}^B) \cdot \Lambda_f(\mathcal{M}^C, \mathcal{M}^D)\,\big) \tag{9}$$

in the definition of the $N = 4$ measure, then the number of unlifted Grassmann modes falls from $16n$ to 16. As discussed in [10], these 16 are precisely the modes generated by the the eight supersymmetric and eight superconformal symmetries of the Lagrangian which are broken by the instanton.

Despite the disappearance of the anomaly in the $N = 4$ case, the remaining properties (i)-(iv) are highly restrictive, and we believe the Ansatz (7) to be unique. As further nontrivial checks, we will also compare this proposed measure against the known first-principles expression in the 2-instanton sector [11,8]. Moreover we will derive, and verify, a stringent RG relation between the measures (1), (4) and (7). This will also serve to relate the 1-instanton factors C_1, C'_1 and C''_1.

3. Supersymmetric invariance of the $N = 4$ measure

The proof of properties (ii), (iii), and (iv) proceeds just as for the $N = 1$ and $N = 2$ cases in (I), and need not be repeated here. In particular, the cluster condition (iii) fixes the overall n-instanton constant in (7) in terms of the 1-instanton factor C''_1, as before. Here we need only focus on property (i), invariance under $N = 4$ supersymmetry. Let us recall how this was established in (I) in the $N = 1$ and $N = 2$ cases. Under an infinitesimal $N = 1$ supersymmetry transformation $\bar\xi\bar Q + \xi Q$, one has [12,13] $\delta a_{\alpha\dot\alpha} = \bar\xi_{\dot\alpha}\mathcal{M}_\alpha$ and $\delta\mathcal{M}_\alpha = -4ib\xi_\alpha$, with b as in (I). Thus the argument of the second δ-function in (1) is invariant, while the argument of the first δ-function picks up an admixture of the second. It follows that the product of the δ-functions is an $N = 1$ invariant.

Next we turn to the $N = 2$ measure (4). The trick here is to represent $(\det \mathbf{L})^{-1}$ as an integral:[2]

$$(\det \mathbf{L})^{-1} \;=\; \int \prod_{\langle ij\rangle_n} d\mathcal{A}_{\rm tot}(\mathcal{M},\mathcal{N})_{i,j}\; \delta\big(\,(\mathbf{L}\cdot \mathcal{A}_{\rm tot}(\mathcal{M},\mathcal{N}) - \Lambda_f(\mathcal{M},\mathcal{N}))_{i,j}\,\big)\;, \tag{10}$$

[2] Here, and in the $N = 2$ and $N = 4$ supersymmetry algebras to follow, we set the adjoint Higgs VEV to zero for simplicity. This suffices for the collective coordinate integration measure, which is necessarily the same with or without a VEV as noted above. When the VEV vanishes, $\mathcal{A}_{\rm tot}(\mathcal{M},\mathcal{N}) \equiv \mathcal{A}' + \mathcal{A}'_f(\mathcal{M},\mathcal{N})$ collapses to the fermion bilinear $\mathcal{A}'_f(\mathcal{M},\mathcal{N})$.

with $\mathcal{A}_{\rm tot}$ and Λ_f as in (I). Now consider the behavior of the arguments of the four δ-functions, respectively the three in Eq. (4) and the one in Eq. (10), under an infinitesimal $N=2$ transformation $\bar\xi_1\bar Q_1+\bar\xi_2\bar Q_2+\xi_1Q_1+\xi_2Q_2$. Recall the action of the $N=2$ supersymmetry on the collective coordinates [12]:

$$\begin{aligned}
\delta a_{\alpha\dot\alpha} &= \bar\xi_{1\dot\alpha}\mathcal{M}_\alpha+\bar\xi_{2\dot\alpha}\mathcal{N}_\alpha &(11a)\\
\delta\mathcal{M}_\gamma &= -4ib\xi_{1\gamma}-2\sqrt{2}\,\mathcal{C}_{\gamma\dot\alpha}(\mathcal{M},\mathcal{N})\,\bar\xi_2^{\dot\alpha} &(11b)\\
\delta\mathcal{N}_\gamma &= -4ib\xi_{2\gamma}+2\sqrt{2}\,\mathcal{C}_{\gamma\dot\alpha}(\mathcal{M},\mathcal{N})\,\bar\xi_1^{\dot\alpha} &(11c)\\
\delta\mathcal{A}_{\rm tot}(\mathcal{M},\mathcal{N}) &= 0 &(11d)
\end{aligned}$$

Here $\mathcal{C}_{\gamma\dot\alpha}(\mathcal{M},\mathcal{N})$ is the $(n+1)\times n$ quaternion-valued matrix

$$\mathcal{C}(\mathcal{M},\mathcal{N}) \;=\; \begin{pmatrix} -w_k\mathcal{A}_{\rm tot}(\mathcal{M},\mathcal{N})_{k1} & \cdots & -w_k\mathcal{A}_{\rm tot}(\mathcal{M},\mathcal{N})_{kn} \\ & \big[\,\mathcal{A}_{\rm tot}(\mathcal{M},\mathcal{N})\,,\,a'\,\big] & \end{pmatrix}\,. \tag{12}$$

Thus $\mathcal{A}_{\rm tot}(\mathcal{M},\mathcal{N})$, in addition to being the dummy of integration in Eq. (10), also completes the $N=2$ algebra; see (I) for a review of its relation to the adjoint Higgs. Using Eq. (11), it is straightforward to show that, under $\bar\xi_1\bar Q_1$, the arguments of the four δ-functions transform into linear combinations of one another, as follows. The argument of the second δ-function is invariant; the arguments of the first and fourth pick up an admixture of the second; and the argument of the third picks up an admixture of the fourth. (Under $\bar\xi_2\bar Q_2$, exchange the roles of the second and third δ-functions; also ξ_1Q_1 and ξ_2Q_2 act trivially.) Clearly the superdeterminant of such an "upper triangular" linear transformation is unity; it follows that the product of the δ-functions is indeed an $N=2$ invariant.

Lastly we turn to the $N=4$ measure, Eq. (7). We wish to demonstrate that it is invariant under an $N=4$ supersymmetry transformation $\sum_{A=1}^4\bar\xi_A\bar Q_A+\xi_AQ_A$. The obvious extension of Eq. (11) to the $N=4$ case, consistent with $SU(4)_R$ symmetry, reads:

$$\begin{aligned}
\delta a_{\alpha\dot\alpha} &= \sum_{A=1,2,3,4}\bar\xi_{A\dot\alpha}\mathcal{M}^A_\alpha &(13a)\\
\delta\mathcal{M}^A_\gamma &= -4ib\xi_{A\gamma}-2\sqrt{2}\sum_{B=1,2,3,4}\mathcal{C}_{\gamma\dot\alpha}(\mathcal{M}^A,\mathcal{M}^B)\,\bar\xi_B^{\dot\alpha} &(13b)\\
\delta_A\mathcal{A}_{\rm tot}(\mathcal{M}^A,\mathcal{M}^B) &= \delta_B\mathcal{A}_{\rm tot}(\mathcal{M}^A,\mathcal{M}^B) \;=\; 0 &(13c)
\end{aligned}$$

where δ_A denotes a variation under $\bar\xi_A\bar Q_A+\xi_AQ_A$ only (no sum on A). Here we are using the fact that in the absence of a VEV both $\mathcal{A}_{\rm tot}$ and $\mathcal{C}$ are antisymmetric in their fermionic

arguments: $\mathcal{C}(\mathcal{M}^A,\mathcal{M}^B) = -\mathcal{C}(\mathcal{M}^B,\mathcal{M}^A)$ and $\mathcal{C}(\mathcal{M}^A,\mathcal{M}^A) = 0$. Note that Eq. (13$b$) encompasses not only Eqs. (11b,c) given above, but also the $N=2$ transformation law for the adjoint Higgsinos $\mathcal{M}^3$ and $\mathcal{M}^4$; see Eq. (12) of Ref. [10]. The $N=4$ algebra (13) is completed by giving the (nonvanishing) transformation law for $\delta_C\mathcal{A}_{\rm tot}(\mathcal{M}^A,\mathcal{M}^B)$ where A, B and C are distinct. This cumbersome expression is straightforward to derive from the defining equation for $\mathcal{A}_{\rm tot}$ [8], but is not actually needed below. Finally we note that this $N=4$ algebra, like the $N=2$ algebra (11), is correct only equivariantly, up to transformations by the internal $O(n)$ group [12]. Consequently it should only be applied to $O(n)$ singlets, which suffices for present purposes (see property (ii) above).

Using Eq. (13), we can now check that the proposed measure (7) is indeed an $N=4$ invariant. For concreteness let us focus (say) on the fourth supersymmetry, $\bar\xi_4\bar Q_4$. As in the $N=2$ case, we introduce an integral representation for $(\det\mathbf{L})^{-3}$:

$$(\det\mathbf{L})^{-3} \;=\; \int\prod_{A\neq 4}\prod_{\langle ij\rangle_n} d\mathcal{A}_{\rm tot}(\mathcal{M}^A,\mathcal{M}^4)_{i,j}\;\delta\big(\,(\mathbf{L}\cdot\mathcal{A}_{\rm tot}(\mathcal{M}^A,\mathcal{M}^4)-\Lambda_f(\mathcal{M}^A,\mathcal{M}^4))_{i,j}\,\big)\;. \tag{14}$$

The index A ranges over the three supersymmetries orthogonal to the one under examination, in this case $A=1,2,3$. Under $\bar\xi_4\bar Q_4$, the arguments of the first δ-function in Eq. (7) and of the three δ-functions in Eq. (14) gain an admixture of the fermionic constraint $(\bar a\mathcal{M}^4)_{i,j}-(\bar a\mathcal{M}^4)_{j,i}$, which is itself invariant as per Eq. (13b). Likewise the arguments of the remaining three fermionic δ-functions, namely $(\bar a\mathcal{M}^A)_{i,j}-(\bar a\mathcal{M}^A)_{j,i}$ with $A=1,2,3$, gain admixtures of the arguments of the three corresponding δ-functions in (14). So, once again, the linear transformation has an upper-triangular structure with superdeterminant unity, implying that the product of all the δ-functions in Eqs. (7) and (14) is invariant under $\bar\xi_4\bar Q_4$. Invariance of the measure (7) under the other three $\bar\xi_A\bar Q_A$ follows by permuting the indices in the above discussion, whereas the $\xi_A Q_A$ act trivially as before.

4. Two-instanton check of the $N=4$ measure

As a first nontrivial consistency check of our proposed $N=4$ measure (7), let us show that it agrees with the known first-principles measure in the 2-instanton sector [11,8]. The discussion exactly parallels that of Sec. 3.5 of (I) for the $N=2$ case, except that the fermionic zero-mode Jacobian $J_{\rm fermi}$ should be squared, there being twice as many adjoint fermion zero modes in the $N=4$ model. Consequently one has [11,8]:

$$\begin{aligned}\int d\mu^{(2)}_{\rm phys} \;&=\; \frac{(C_1'')^2}{\mathcal{S}_2}\int d^4w_1d^4w_2d^4a'_{11}d^4a'_{22}\prod_{A=1,2,3,4} d^2\mu_1^Ad^2\mu_2^Ad^2\mathcal{M}^{A\prime}_{11}d^2\mathcal{M}^{A\prime}_{22}\\ &\qquad\times\;\frac{64|a_3|^4}{H^3}\,\Big|\,|a_3|^2\,-\,|a'_{12}|^2\,-\,\tfrac18\frac{d\Sigma^\phi}{d\phi}\big|_{\phi=0}\,\Big|\;,\end{aligned}\tag{15}$$

in the notation of (I). Like Eqs. (2.55) and (3.28) in (I), this is a "gauge fixed" measure, the form of which explicitly breaks both $O(2)$ and supersymmetry invariance. Also as in (I), the overall factor in Eq. (15) is tied to 't Hooft's 1-instanton factor C_1'' by cluster decomposition.[3] Inserting the factors of unity

$$1 = 16|a_3|^4 \int d^4a'_{12} \prod_{c=1,2,3} \delta\big(\tfrac{1}{4}\mathrm{tr}_2\,\tau^c[(\bar{a}a)_{1,2}-(\bar{a}a)_{2,1}]\big)\delta\big(\bar{a}_3a'_{12}+\bar{a}'_{12}a_3-\tfrac{1}{2}\Sigma(a_3,a_0,w_1,w_2)\big) \tag{16}$$

and

$$1 = \prod_{A=1,2,3,4} \frac{1}{4|a_3|^2} \int d^2\mathcal{M}^{A\prime}_{12}\,\delta^2\big((\bar{a}\mathcal{M}^A)_{1,2} - (\bar{a}\mathcal{M}^A)_{2,1}\big) \tag{17}$$

into Eq. (15), performing the change of dummy integration variables $a \to a^\phi$ and $\mathcal{M}^A \to \mathcal{M}^{A\phi}$ described in (I), inserting a final factor of unity $1 = (2\pi)^{-1}\int_0^{2\pi} d\phi$, and recalling that in the 2-instanton sector $\det \mathbf{L} = H$, one readily recovers the $O(2)$- and $N=4$ invariant form for the measure, Eq. (7). See Secs. 2.5 and 3.5 of (I) for calculational details.

5. RG relation between the $N=1$, $N=2$, and $N=4$ measures

A distinct consistency check is to invoke the physical requirement of RG decoupling to relate the $N=1$, $N=2$ and $N=4$ measures. Let us focus first on the $N=1$ and $N=2$ measures, Eqs. (1) and (4). In the language of $N=1$ superfields, the particle content of $N=2$ supersymmetric Yang-Mills theory consists of a gauge superfield $V=(v_m,\lambda_\alpha)$ and an adjoint chiral superfield $\Phi=(A,\psi_\alpha)$. Let us add a mass term $m\,\mathrm{tr}_2\,\Phi^2\big|_{\theta^2}+\mathrm{H.c.}$ for the matter superfield, breaking the $N=2$ supersymmetry down to $N=1$. To leading semiclassical approximation, this is equivalent to inserting a Higgsino mass factor [10]

$$\exp\big(-m\pi^2\,\mathrm{Tr}_n\,\mathcal{N}^{\gamma T}(\mathcal{P}_\infty+1)\mathcal{N}_\gamma\big) \tag{18}$$

into the integrand of Eq. (4). Here $\mathcal{P}_\infty$ is the $(n+1)\times(n+1)$ matrix $\mathrm{diag}(1,0,\cdots,0)$ in the conventions of [11]. (There are bosonic mass terms too but their effect on the semiclassical physics is down by one factor of the coupling as they require the elimination of an auxiliary F field.) RG decoupling means that in the double scaling limit for the mass and coupling constant, defined by $m\to\infty$ and $g\to 0$ with a certain combination held fixed (namely, the left-hand side of Eq. (23b) below), the Φ multiplet decouples from

[3] In the clustering limit taken in (I), $H\to 4|a_3|^2\to\infty$, which accounts for the factor of 64 in Eq. (15). Similarly, the right-hand side of Eq. (3.29) of (I) should contain a factor of 4.

the physics. Concomitantly, the $N=2$ measure should collapse to the $N=1$ measure. Comparing Eqs. (1) and (4) then leads to the RG consistency requirement in this limit:

$$\begin{aligned}(C_1')^n \int \prod_{i=1}^n d^2\nu_i \prod_{(ij)_n} d^2\mathcal{N}'_{ij} \prod_{\langle ij\rangle_n} \delta^2\big((\bar a\mathcal{N})_{i,j} - (\bar a\mathcal{N})_{j,i}\big)\, \exp\big(-m\pi^2\,\mathrm{Tr}_n\,\mathcal{N}^{\gamma T}(\mathcal{P}_\infty+1)\mathcal{N}_\gamma\big) \\ \longrightarrow\ (C_1)^n\cdot \det \mathbf{L}\ .\end{aligned} \tag{19}$$

Note that each side of Eq. (19) is a complicated function of the bosonic moduli a, and it is far from obvious that the two sides are proportional to one another. We therefore regard this relation as a stringent consistency check on our proposed integration measures. To establish this proportionality, we examine the integral

$$\begin{aligned}\mathcal{I}_n(a;m) \ =\ & \int \prod_{i=1}^n d^2\nu_i \prod_{(ij)_n} d^2\mathcal{N}'_{ij} \prod_{\langle ij\rangle_n} d\mathcal{A}_{\rm tot}(\mathcal{M},\mathcal{N})_{i,j} \\ & \times \prod_{\langle ij\rangle_n} \delta^2\big((\bar a\mathcal{N})_{i,j} - (\bar a\mathcal{N})_{j,i}\big)\, \delta\big(\,(\mathbf{L}\cdot\mathcal{A}_{\rm tot}(\mathcal{M},\mathcal{N}) - \Lambda_f(\mathcal{M},\mathcal{N}))_{i,j}\,\big) \\ & \times\ \exp\big(-m\pi^2\,\mathrm{Tr}_n\,\mathcal{N}^{\gamma T}(\mathcal{P}_\infty+1)\mathcal{N}_\gamma\big)\ .\end{aligned} \tag{20}$$

On the one hand, by construction, $\mathcal{I}_n$ is a function of the bosonic collective coordinates a only. On the other hand, from Eq. (11), one confirms that $\mathcal{I}_n$ is an $N=1$ invariant under $\bar\xi_1\bar Q_1+\xi_1 Q_1$.[4] But, as mentioned earlier, the only purely bosonic supersymmetry invariants are constants! Thus $\mathcal{I}_n(a;m)\equiv\mathcal{I}_n(m)$, independent of the matrix a. Performing the $\mathcal{A}_{\rm tot}$ integration in Eq. (20) using Eq. (10), and comparing to Eq. (19), establishes the claimed proportionality, and leads to the condition

$$\mathcal{I}_n(m) \ =\ \lim\ \big(C_1/C_1'\big)^n \tag{21}$$

where the right-hand side is understood in the double scaling limit.

In order to evaluate $\mathcal{I}_n$ explicitly, and thereby relate C_1 and C_1', it suffices to choose a such that each side of Eq. (19) is nonzero. A convenient such choice, consistent with the ADHM constraints $\bar a a = (\bar a a)^T$, is $a'_{\alpha\dot\alpha} = \mathrm{diag}(a'_{11\,\alpha\dot\alpha},\cdots,a'_{nn\,\alpha\dot\alpha})$ and $w_{k\,\alpha\dot\alpha}=0$, in the notation of Eq. (2). For this choice, the eigenvectors of $\mathbf{L}$ are the $n\times n$ antisymmetric matrices t_{ij}, $1\le i<j\le n$, defined by their matrix elements $\big(t_{ij}\big)_{kl} = \delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}$. From the definition of $\mathbf{L}$ (see Eq. (3.10) of (I)) one sees that $\mathbf{L}\cdot t_{ij} = |a'_{ii}-a'_{jj}|^2\cdot t_{ij}$, so that the right-hand side of Eq. (19) is simply $(C_1)^n\cdot\prod_{\langle ij\rangle_n}|a'_{ii}-a'_{jj}|^2$. On the other

[4] This claim is only true if $\mathcal{M}$ satisfies the super-ADHM relation $\bar a\mathcal{M} = (\bar a\mathcal{M})^T$ [7], but this suffices for our needs, since this relation is enforced by δ-functions in both Eqs. (1) and (4).

hand, the left-hand side of Eq. (19) is trivially evaluated for this choice of a. The ν_i and $\mathcal{N}'_{ii}$ integrations are saturated solely by the mass term, giving $(-2m\pi^2)^n$ and $(-m\pi^2)^n$, respectively, while the $\mathcal{N}'_{ij}$ integrations with $1 \le i < j \le n$ are saturated by the δ-functions, and yield $\prod_{\langle ij\rangle_n} |a'_{ii} - a'_{jj}|^2$. So the a dependence indeed cancels out as claimed, leaving

$$\mathcal{I}_n(m) \;=\; (2m^2\pi^4)^n \; . \tag{22}$$

Equivalently, in the double scaling limit, from Eqs. (21), (3) and (5):

$$C_1 \;=\; \lim\, 2m^2\pi^4 C'_1 \; , \tag{23a}$$

$$\Lambda^6_{N=1} \;=\; \lim\, m^2 \Lambda^4_{N=2} \; . \tag{23b}$$

By identical arguments, one can also flow from the $N = 4$ measure (7) to the $N = 2$ measure (4), by inserting the $N = 2$ invariant Higgsino mass term [10]

$$\exp\big(-2im\pi^2 \mathrm{Tr}_n\, \mathcal{M}^{4T}(\mathcal{P}_\infty + 1)\mathcal{M}^3\big) \tag{24}$$

into the integrand of the former, and carrying out the integrations over $\mathcal{M}^3$ and $\mathcal{M}^4$. This integration is proportional to $(\det \mathbf{L})^2$ as required, and yields the relation

$$C'_1 \;=\; \lim\, 4m^4\pi^8 C''_1 \tag{25a}$$

or equivalently

$$\Lambda^4_{N=2} \;=\; \lim\, m^4 \exp(-8\pi^2/g^2_{N=4}) \tag{25b}$$

in the double scaling limit. This result holds regardless of whether the expression (9) is included in the definition of the $N = 4$ measure, as this term is subleading compared to the mass term (24) in this limit.

Note that Eqs. (23b) and (25b) are consistent with the standard prescriptions in the literature for the RG matching of a low-energy and a high-energy theory [14,15]. The absence of numerical factors on the right-hand sides of these relations reflects the absence of threshold corrections in the PV scheme.

6. The classical $N = 0$ measure

Finally, we can flow from the $N = 1$ measure to the purely bosonic "$N = 0$" measure as well, by inserting a gaugino mass factor

$$\exp\big(- m\pi^2 \, \mathrm{Tr}_n \, \mathcal{M}^{\gamma T} (\mathcal{P}_\infty + 1) \mathcal{M}_\gamma \big) \tag{26}$$

into the integrand of Eq. (1), and carrying out the $\mathcal{M}$ integration using the above identities. In this way one finds for the $N = 0$ measure the $O(n)$ invariant expression:

$$\begin{aligned} \int d\mu_{\rm cl}^{(n)} \ &= \frac{(C_1^{(0)})^n}{\mathrm{Vol}\big(O(n)\big)} \int \prod_{i=1}^n d^4 w_i \prod_{(ij)_n} d^4 a'_{ij} \\ &\times \prod_{\langle ij \rangle_n} \prod_{c=1,2,3} \delta\big(\tfrac{1}{4} \mathrm{tr}_2 \, \tau^c [(\bar a a)_{i,j} - (\bar a a)_{j,i}] \big) \ \det \mathbf{L} \ , \end{aligned} \tag{27}$$

where $C_1^{(0)}$ is related to C_1 in the double scaling limit via

$$C_1^{(0)} \ = \ \lim 2 m^2 \pi^4 C_1 \ = \ \lim 2^{10} \, m^2 \pi^4 \, \Lambda_{N=1}^6 \ . \tag{28}$$

(In the 1-instanton case, the second line in Eq. (27) is absent as always.) This expression may be seen to obey cluster decomposition by the same arguments as for the $N = 1$ and $N = 2$ measures (see Secs. 2.4 and 3.4 of (I)); in particular it is convenient for this purpose to represent $\det \mathbf{L}$ as a Grassmann integral analogous to the bosonic representation (10) for $(\det \mathbf{L})^{-1}$. As a check of Eq. (27), in the 2-instanton sector it can be shown to be equivalent to the first-principles (but $O(2)$-breaking) form for the classical measure written down by Osborn [11]. The proof of this equivalence is identical to that for the $N = 1$ and $N = 2$ cases considered in Secs. 2.5 and 3.5 of (I) and to the $N = 4$ case discussed above, and is left to the reader.

As stated earlier, Eq. (27) is purely a classical measure since it neglects the one-loop small-fluctuations 't Hooft determinants over positive-frequency modes in the self-dual background. While such classical collective coordinate integration measures have been studied before in the ADHM problem, at the 2-instanton level [16,11], they are more familiar in the context of BPS multi-monopoles [17,18]. There they correspond to volume forms obtained by taking the appropriate power of the classical metric 2-forms. The classical hyper-Kähler metric on the multi-monopole moduli space is physically important, as it governs the nonrelativistic scattering of BPS monopoles in (3+1)-dimensions. In the ADHM case the corresponding classical hyper-Kähler metric is presently unknown (although the classical volume form (27) may provide a useful constraint). By analogy with monopoles, such a metric would govern the nonrelativistic scattering of four-dimensional

multi-instantons embedded in a *five*-dimensional space-time, a rather arcane physical problem.

Assuming, instead, that one is primarily interested in the contribution of ADHM multi-instantons to four-dimensional physics, knowledge of the classical integration measure does not suffice. One must know the one-loop determinants as well, for two compelling reasons. First, these determinants enter the semiclassical expansion at order g^0, just like Eq. (27) itself. Second, without them, Green's functions turn out to be scale- and scheme-dependent, hence unphysical. In particular, the classical 1-instanton factor $C_1^{(0)}$ in the PV scheme may be extracted from Eq. (12.1) of [3]:

$$C_1^{(0)}(\mu_0) \;=\; 2^{10}\,\pi^4\,\mu_0^8\,\exp\big(-8\pi^2/g^2_{{}_{N=0}}(\mu_0)\big) \;=\; 2^{10}\,\pi^4\,\mu_0^8\,\big(\Lambda_{{}_{N=0}}/\mu_0\big)^{22/3} \tag{29}$$

which explicitly depends on the subtraction scale μ_0. (Presumably Eq. (28) above should be understood at the matching scale, $\mu_0 = m$.) Only when the one-loop determinant is included, giving a contribution [3]

$$(\mu_0\,|w|)^{-2/3}\,e^{-\alpha(1)}\;, \quad \alpha(1) \cong 0.443307 \tag{30}$$

with $|w|$ the instanton size, do the explicit factors of μ_0 in Eq. (29) cancel out, leaving a scale-independent, RG-invariant answer for the physical 1-instanton measure. To date, there has been substantial progress towards the calculation of these one-loop determinants in the general ADHM background [5,6], based on the expressions for the Green's functions derived in [7]. Taken together with the classical measure (27) above, complete knowledge of these determinants would ultimately put ADHM multi-instantons on the same solid field-theoretic footing as single instantons have been since the work of 't Hooft.

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